YET MORE ON THE LINEAR SEARCH PROBLEM*

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ABSTRACT

The linear search problem has been discussed previously by one of the present authors. In this paper, the probability distribution of the point sought in the real line is not known to the searcher. Since there is no *a priori* choice of distribution which recommends itself above all others, we treat the situation as a game and obtain minimax type solutions. Different minimaxima apply depending on the factors which one wishes to minimize (resp. maximize). Certain criteria are developed which help the reader judge whether the results obtained can be considered "good advice" in the solution of real problems analogous to this one.

1. Introduction.

The linear search problem can be expressed in intuitive terms as follows: consider the real line as a highway, on which the searcher (a man in an automobile) and a goal (some object being sought) are located. The automobile has a certain fixed speed, and it is the purpose of the searcher to find the object as soon as possible. The problem is to devise a reasonable procedure for the searcher to follow.

Of course, the problem as here formulated is not yet a mathematical one, for we have no criteria for the measurement of how well any chosen procedure accomplishes its object. To do this, we imagine that the goal is located along the line according to a (known or unknown) probability distribution F. Then we want that search procedure which minimizes the expected time (or expected path length) to reach the goal, assuming the searcher starts at 0.

In [1] and [2], some study was made of the situation for a known distribution

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F, and certain theorems show conditions under which the minimum expected time is actually achievable. However, no algorithm has been found for extracting the minimizing search procedure.

In this paper, we will take up the companion problem concerning an unknown distribution F. Just as there is no natural choice for F itself, so there is no natural distribution of such distributions. We will treat the matter as a game between the searcher and a hider, who chooses a distribution so as to maximally frustrate search. Thus the problem is modified to fall within Game Theory criteria, and we seek minimaxima, etc., as a way of offering "advice" to the searcher.

In this work, an algorithm of sorts is exhibited for the modified problem. Whether this algorithm answers the question is a matter of taste, and the reader is left to give his own judgement.

In each case, if the target point is put on the real line according to a probability distribution F and sought for by a generalized search procedure $x = \{x_i\}_{i=-\infty}^{+\infty}$ (see definitions in [2]), then the distance traveled from 0 to the point t, which is designated as $X(x,t)$ must be integrated to yield the *expected length of path* $X(x) = \int_{-\infty}^{+\infty} X(x,t)dF(t)$. This quantity is the "payoff" of the game we mentioned earlier, and it will be the purpose of the searcher to minimize it, and of his opponent to maximize it by a nasty choice of F.

It is easily seen that $X(x,t) \geqq |t|$ for every search procedure x, so that $X(x) \geqq \int_{-\infty}^{+\infty} |t| dF(t) = M_1(F)$. On the other hand, if we define $x(\alpha,\delta)$ as the procedure whose ith entry is $-\delta(-\alpha)^i$, $\forall -\infty < i < +\infty$, then $X(x(2,\delta)) < 9M_1$, for all choices of δ. Since the efficiency of any strategy is thus reasonably measured against the size of M_1, let us assume that $M_1 = 1$. We wish to know for each F what is the value of $M_0(F) = \inf_x X(x)$, and we also want to know $\max_F M_0(F)$. Further, we wish to find ways of achieving or approximating these maxima and minima.

2. How good is 9 as an estimate?

Our first task is to verify that

1. LEMMA. *For every $\delta > 0$ and every $-\infty < t < +\infty$, $X(x(2,\delta),t) < 9|t|$.*

PROOF: Assume first that $t > 0$, and let n be so chosen that $2^n\delta < t \leqq 2^{n+2}\delta$. Then $X(x(2,\delta),t) = \sum_{i=-\infty}^{n+1} \delta 2 \cdot 2^i + |t| = \delta 2^{n+3} + |t| = 8\delta 2^n + |t| < 9|t|$. If $t < 0$, choose n so that $-2^{n+2}\delta \leqq t \leqq -2^n\delta$ and make the same analysis. Q.E.D.

Now we show converse to this lemma:

2. LEMMA. *For every search procedure x, and each $\varepsilon > 0$, we can find a distribution F with $M_1(F) = 1$ so that $X(x) > 9 - \varepsilon$.*

PROOF. To prove this lemma, we define for each search procedure X the number $r_1(x) = \sup_{|t| \geqq 1}(|t|^{-1}X(x,t))$. We will show that $r_1(x) \geqq 9$, and this will in turn imply our conclusion, as we shall see below.

To show that $r_1(x) \geqq 9$, let $x = \{\cdots x_{-1}, x_0, x_1, x_2, \cdots\}$, where the numbering is so chosen that x_1 is the first entry greater than 1. We can assume that x is a strong search plan, i.e. that $x_n = x_{n+2} \Rightarrow x_n = x_{n+1} = 0$ (see [2]), since otherwise we could replace x by a search plan which, for eve y distribution F, has a lower value of $X(x)$, by simply eliminating x_n and x_{n+1}.

For every $n \geqq 1$, we have $x_n \neq x_{n+2}$. If n is large enough, say $n > N$, we have $|x_n| \geqq 1$. Then for t between x_n and x_{n+2}, we have

$$X(x,t) = 2\sum_{i=-\infty}^{n+1} |x_i| + |t| \leqq r_1(x)|t|,$$

so that

$$\sum_{i=-\infty}^{n+1} |x_i| \leqq \tfrac{1}{2}(r_1(x)-1)|t| \leqq \tfrac{1}{2}(r_1(x)-1)|x_n|.$$

We assume, contrary to the conclusion, that $r_1(x) < 9$, so that $\frac{1}{2}(r_1(x) - 1) < 4$. Denote $\frac{1}{2}(r_1(x) - 1)$ as $\gamma = 4 - \beta$, and set

$$y_n = \frac{1}{2^n}\sum_{i=-\infty}^{n+1} |x_n|.$$

Then $y_n > 0$ and the condition

$$\sum_{i=-\infty}^{n+1} |x_i| \leqq \gamma|x_n|$$

gives

$$\begin{aligned}
2^{n+2}\tfrac{1}{2}(y_{n-1} + y_{n+1}) &= 4\sum_{i=-\infty}^{n} |x_i| + \sum_{i=-\infty}^{n+2} |x_i| \\
&\leqq 4\sum_{i=-\infty}^{n} |x_i| + \gamma|x_{n+1}| \\
&= 4\sum_{i=-\infty}^{n+1} |x_i| - \beta|x_{n+1}| \\
&\leqq 4\cdot 2^n y_n - \gamma^{-1}\beta\sum_{i=-\infty}^{n+2} |x_i| \\
&= 2^{n+2}y_n - 2^{n+1}\gamma^{-1}\beta y_{n+1},
\end{aligned}$$

so that

$$\tfrac{1}{2}(y_{n-1} + y_{n+1}) \leqq y_n - \tfrac{1}{2}\gamma^{-1}\beta y_{n+1}$$
$$\leqq y_n.$$

Thus, $\{y_n\}$ is a concave positive sequence, and if it is bounded away from 0 for n large enough, say by $\eta > 0$, then $\frac{1}{2}(y_{n+1} - y_n) < \frac{1}{2}(y_n - y_{n-1}) - \frac{1}{2}\gamma^{-1}\beta\eta$, so that $\frac{1}{2}(y_{n+1} - y_n) < 0$ for n sufficiently large, and thus $y_n < 0$ for n sufficiently large, contrary to hypothesis. But a positive concave sequence cannot have arbitrarily small terms. Thus we have reached a contradiction and the hypothesis $\gamma < 4$ must be abandoned.

Now let a number t_0 be chosen with $|t_0| > 1$ and $|t_0^{-1}| X(x,t_0) > 9 - \varepsilon$. Let

$$F(t) = 0, \qquad \forall t < 0$$
$$= 1 - |t_0|^{-1}, \ \forall 0 \leqq t < |t_0|$$
$$= 1, \qquad \forall |t_0| \leqq t.$$

Then

$$\int_{-\infty}^{+\infty} |t| \, dF(t) = 1, \ \text{and} \ \int_{-\infty}^{+\infty} X(x,t)dF(t) = |t_0|^{-1}X(x,t_0) > 9 - \varepsilon. \qquad \text{Q.E.D.}$$

3. Lemma. *If x is so chosen that for every distribution F with $M_1(F) = 1$, we have $X(x) \leqq 9$, then*

$$x_n = x_0(-2)^n, \ \forall -\infty < n < +\infty.$$

Proof. We know from the proof of Lemma 2 that $r_1(x) \geqq 9$. In fact for every $a \geqq 0$, we can define $r_a(x) = \sup_{|t|>a} |t|^{-1}X(x,t)$. Then by the same proof, we have $r_a(x) \geqq 9$, $\forall a > 0$, and thus $r_0(x) \geqq 9$ as well. If $r_0(x) = 9$, then $\{y_n\}$, defined as in the proof of Lemma 2, is positive for all $-\infty < n < +\infty$, and is convex in its entire range. It follows at once that $r_0(x) = 9$ implies $\{y_n\}$ is a constant sequence, and $x_n = x_0(-2)^n$.

Suppose that it were possible that $r_a(x) > 9$, $a \geqq 1$. Then choose $t_0 > a \geqq 1$ so that $|t_0|^{-1}X(x,t) > 9$, and define F again as in the last part of the proof of Lemma 2. This will give $X(x) > 9$ and $M_1(F) = 1$. Thus, $r_a(x) \leqq 9$, $\forall a \geqq 1$.

Now suppose that $r_0(x) > 9$. Let t_0 again be chosen so that $|t_0|^{-1}X(x,t_0) > 9$, and set $\varepsilon = X(x,t_0) - 9|t_0| > 0$. We know $|t_0| \leqq 1$, and since $r_2(x) \geqq 9$, we can find a $t_1 > 2$ satisfying $|t_1|^{-1}X(x,t_1) > 9 - \frac{1}{2}\varepsilon$. Let F be a distribution assigning probability $\frac{1}{2}$ to t_0, $|t_1|^{-1}(1-\frac{1}{2}|t_0|)$ to t_1, and the rest (i.e. $1-\frac{1}{2}-|t_1|^{-1}(1-\frac{1}{2}|t_0|)$) to 0. Then $M_1(F)=1$ and

$$\begin{aligned}\int_{-\infty}^{+\infty}(X(x,t)-9|t|)dF(t) &= (X(x,t_0)-9|t_0|)\cdot\tfrac{1}{2}\\ &\quad+(X(x,t_1)-9|t_1|)|t_1|^{-1}(1-\tfrac{1}{2}|t_0|)\\ &> \tfrac{1}{2}\varepsilon-\tfrac{1}{2}\varepsilon|t_1|\cdot|t_1|^{-1}(1-\tfrac{1}{2}|t_0|)\\ &> 0,\end{aligned}$$

so that

$$\int_{-\infty}^{+\infty}X(x,t)dF(t)>9.$$

It follows that $r_0(x)>9$ is impossible, and $r_0(x)=9$, which proves the lemma. Q.E.D.

REMARK. The distributions used as examples in the proofs of Lemmas 2 and 3 are atomic. Actually, a small amount of tinkering could have been done to make them continuous, absolutely continuous, or even C^∞, with the same essential properties.

Thus we see that in choosing a search plan, we can "play it safe" by choosing one of the search plans $x(2,\delta)$. No other strategy is as safe as one of these. However, even among the "safe" strategies, some are better then others, and we will now consider the search problem as a game in which the searcher must choose a search strategy from among the safe ones, while his antagonist chooses any distribution F with $M_1(F)=1$.

4. THEOREM. *If $M_1(F)=1$ and $\alpha>0$, then for at least one value of $\delta>0$ the strategy $x=x(\alpha,\delta)$ yields $X(x)\leqq 1+(\alpha+1)/\ln\alpha$.*

PROOF. We shall show more. By putting a measure on the parameter δ, we shall show that the *average* of $X(x)$ for $x(\alpha,\delta)$ satisfies the given inequality. The result will then follow from the mean value theorem. Let $-\infty<\eta<+\infty$, and $\delta=\alpha^\eta$. Then define $y(\eta)=x(\alpha,\delta)$. We see that the entries of $y(\eta)$ and $y(\eta+2)$ are identical, except that they have been moved over.

Let $Y(\eta,t)=X(y(\eta),t)$. Then for each t, $Y(\eta,t)$ is periodic of period 2. We are interested in $\min_\eta Y(\eta)=X(y(\eta))$. For each $t>0$, $-\infty<\eta<+\infty$, we have $n+\eta<\log_\alpha t\leqq n+\eta+2$, where $n=n(t,\eta)$ is the greatest odd integer less than $\log_\alpha t-\eta$. Thus,

$$Y(\eta,t) = 2\sum_{i=-\infty}^{n+1} \alpha^{i+\eta} + t = 2\frac{\alpha^{n+\eta+1}}{1-\alpha^{-1}} + |t|,$$

and similarly for $t < 0$, so that Y is an integrable function of η and t together.

We now wish to evaluate

$$\tfrac{1}{2}\int_0^2 Y(\eta)d\eta = \tfrac{1}{2}\int_0^2 \int_{-\infty}^{+\infty} Y(\eta,t)dF(t)d\eta.$$

We invert the order of integration, and observe that since $Y(\eta,t)$ is periodic in η for each t, we have, for each $t > 0$,

$$\int_0^2 Y(\eta,t)d\eta = \int_{\log_\alpha t-1}^{\log_\alpha t+1} Y(\eta,t)d\eta.$$

For all $\log_\alpha t - 1 < \eta < \log_\alpha t + 1$, we have $n(t,\eta) = -1$, so that

$$Y(\eta,t) = 2\left(\frac{\alpha^\eta}{1-\alpha^{-1}}\right) + t$$

and

$$\int_{\log_\alpha t-1}^{\log_\alpha t+1} Y(\eta,t)d\eta = 2\left(\frac{\alpha^\eta}{(1-\alpha^{-1})\ln\alpha}\Bigg]_{\log_\alpha t-1}^{\log_\alpha t+1} + t\right)$$

$$= 2\left(\frac{\alpha+1}{\ln\alpha}t + t\right).$$

Similarly, when $t < 0$, we have

$$Y(\eta,t) = 2\left(\frac{\alpha^{n+\eta+1}}{1-\alpha^{-1}}\right) + |t|$$

whete $n = n(t,\eta)$ is the smallest even integer with $n + \eta < \log_\alpha|t| \leqq n + \eta + 2$. Thus,

$$\int_0^2 Y(\eta,t)d\eta = \int_{\log_\alpha|t|}^{\log_\alpha|t|+2} Y(\eta,t)d\eta = 2|t|\left(\frac{\alpha+1}{\ln\alpha} + 1\right),$$

as before. Thus, we have

$$\begin{aligned}\tfrac{1}{2}\int_0^2 Y(\eta)d\eta &= \tfrac{1}{2}\int_0^2\int_{-\infty}^{+\infty} Y(\eta,t)dF(t)d\eta \\ &= \int_{-\infty}^{+\infty}\tfrac{1}{2}\int_0^2 Y(\eta,t)d\eta dF(t) \\ &= \int_{-\infty}^{+\infty}\left(\frac{\alpha+1}{\ln\alpha}+1\right)|t|\,dF(t) \\ &= \left(\frac{\alpha+1}{\ln\alpha}+1\right) M_1(F) = \frac{\alpha+1}{\ln\alpha}+1,\end{aligned}$$

since $M_1(F) = 1$.

It follows directly that for some value of η, $Y(\eta) = X(y(\eta)) \leqq (\alpha + 1)/\ln\alpha + 1$.
Q.E.D.

5. COROLLARY. *Among the "safe" strategies, there is one (call it χ) for which* $X(\chi) \leqq 1 + 3/\ln 2$.

It happens, however, that $(\alpha + 1)/\ln\alpha$ does not attain its minimum at $\alpha = 2$, but does so for the solution α_0 of the equation $\alpha_0 \ln \alpha_0 = \alpha_0 + 1$. Thus the minimum value of m_0 is no more than

$$1 + \frac{\alpha_0 + 1}{\ln\alpha_0} = 1 + \alpha_0 .$$

Furthermore, the strategies $x(\alpha_0, \delta)$, when mixed in the indicated way, will assure the searcher of an expected outcome of $1 + \alpha_0$. In taking the lowest expectation, however, he risks a loss as high as

$$1 + \frac{2\alpha_0^2}{\alpha_0 - 1} > 9 .$$

We now raise the question of whether there is not a better mixed strategy than that of the $x(\alpha_0, \delta)$ mixed in the indicated way. The answer is "no" as is seen from the following theorem.

6. THEOREM. *For every $\varepsilon > 0$, there exists a probability distribution F with $M_1(F) = 1$ so that for every search plan x, we have*

$$X(x) \geqq 1 - \frac{\varepsilon}{1 + \varepsilon} + \frac{1 + \alpha_0}{1 + \varepsilon} .$$

PROOF. Let $\varepsilon > 0$, and set

$$\omega = \frac{\varepsilon}{2(1 + \varepsilon)}, \quad b = \frac{\varepsilon}{1 + \varepsilon}, \text{ and } B = \frac{\varepsilon e^{1/\varepsilon}}{1 + \varepsilon} .$$

Set

$$\begin{aligned} F(t) &= 0, && \forall\ -\infty < t \leqq -B \\ &= \frac{\omega}{t}, && \forall\ -B < t \leqq -b \\ &= \tfrac{1}{2}, && \forall\ -b \leqq t \leqq b \\ &= 1 - \frac{\omega}{t}, && \forall\ b \leqq t < B \\ &= 1, && \forall\ B \leqq t < +\infty \end{aligned}$$

Since $F'(0) = 0$, it is immediately apparent from Theorem 1 of [1] that there exists a minimizing search procedure x, and the presence of probability atoms at $-B$ and B assures us that the procedure has only finitely many entries. Let $x = (x_1, x_2, x_3, \cdots, x_{n-1}, x_n, x_{n+1})$. Note that $|x_n| = |x_{n+1}| = B$, and that for all $b \leqq t < B$,

$$F(-t) - F(-B) = F(B) - F(t) = \omega/t.$$

We see at once that

$$\begin{aligned} X(x) = M_1(F) + & 2|x_1|\left(\frac{\omega}{|x_1|} + \frac{\omega}{b}\right) \\ & + 2|x_2|\left(\frac{\omega}{|x_2|} + \frac{\omega}{|x_1|}\right) \\ & + \cdots \\ & + 2|x_{n-1}|\left(\frac{\omega}{|x_{n-1}|} + \frac{\omega}{|x_{n-2}|}\right) \\ & + 2|x_n|\frac{\omega}{|x_{n-1}|}. \end{aligned}$$

Since

$$\begin{aligned} M_1(F) &= \int_{-\infty}^{+\infty} |t|\, dF(t) \\ &= \frac{\omega}{B}\cdot B + \int_{-B}^{-b} |t| \cdot \frac{\omega}{t^2} dt + \int_{b}^{B} |t| \cdot \frac{\omega}{t^2} dt + \frac{\omega}{B}\cdot B \\ &= 2\omega + 2\int_b^B \frac{\omega}{t} dt \\ &= 2\omega + 2\omega(\ln B - \ln b) \\ &= 2\frac{\varepsilon}{2(1+\varepsilon)} + 2\frac{\varepsilon}{2(1+\varepsilon)} \cdot \frac{1}{\varepsilon} = 1, \end{aligned}$$

we have

$$X(x) = 1 + 2\omega\left(n - 1 + \frac{|x_1|}{b} + \frac{|x_2|}{|x_1|} + \cdots + \frac{|x_n|}{|x_{n-1}|}\right).$$

Since

$$\frac{1}{n}\left(\frac{|x_1|}{b}+\frac{|x_2|}{|x_1|}+\cdots+\frac{|x_n|}{|x_{n-1}|}\right) \geqq \left(\frac{|x_1|}{b}\cdot\frac{|x_2|}{|x_1|}\cdots\frac{|x_n|}{|x_{n-1}|}\right)^{1/n}$$

$$= \left(\frac{B}{b}\right)^{1/n} = e^{1/\varepsilon n},$$

$$X(x) \geqq 1 + 2\omega(n - 1 + ne^{1/\varepsilon n})$$

$$= 1 - 2\omega + 2\omega n(1 + e^{1/\varepsilon n})$$

$$= 1 - \frac{\varepsilon}{1+\varepsilon} + \frac{1}{1+\varepsilon}\cdot n\varepsilon(1 + e^{1/\varepsilon n}).$$

The definition of α_0 gives us

$$n\varepsilon(1 + e^{1/\varepsilon n}) \geqq \inf_t \frac{1}{t}(1 + e^t)$$

$$= \inf_s \frac{1}{\ln s}(1 + s) = 1 + \alpha_0,$$

so that

$$X(x) \geqq 1 - \frac{\varepsilon}{1+\varepsilon} + \frac{1+\alpha_0}{1+\varepsilon}. \qquad \text{Q.E.D.}$$

Now we have shown that $\max_F m_0(F) = 1 + \alpha_0$, and have exhibited a mixed strategy whereby the seeker can obtain this minimax. Is there also a probability distribution F which maximizes $m_0(F)$?

7. THEOREM. *For every probability distribution* F, $m_0(F) < 1 + \alpha_0$.

PROOF: We may assume, of course, that F displays no mass point at the origin. We already know that for some η the sequence $y = \{\alpha_0^{k+\eta}\}_{-\infty}^{\infty}$ satisfies $X(y) \leqq 1 + \alpha_0$. Choose N so that

$$\Delta = F(\alpha_0^{-2N+\eta}) - F(\alpha_0^{-2N+1+\eta}) < \frac{1}{\alpha_0}$$

and consider the truncated sequence $\bar{y} = \{\alpha_0^{k+\eta}\}_{-2N}^{\infty}$. We will show that $X(\bar{y}) < X(y)$ thus completing the proof. We have, namely,

$$X(\bar{y},t) \leqq X(y,t) \qquad \text{for} \quad 0 \leqq t \leqq \alpha_0^{-2N+\eta}$$

$$X(\bar{y},t) \leqq 2\alpha_0^{-2N+\eta} + X(y,t) \qquad \text{for} \quad -\alpha_0^{-2N+\eta+1} \leqq t \leqq 0$$

$$X(\bar{y},t) \leqq X(y,t) - 2\sum_{-\infty}^{-2(N+1)} \alpha_0^{k+\eta} \quad \text{for} \quad t \notin [-\alpha^{-2N+\eta+1}, \alpha^{-2N+\eta}].$$

Integrating $dF(t)$, recalling the definition of Δ and using

$$\sum_{k=-\infty}^{-(2N+1)} \alpha_0^{k+\eta} = \frac{\alpha_0^{-2N+\eta}}{\alpha_0 - 1}$$

we obtain

$$X(\bar{y}) \leqq 2\alpha_0^{-2N+\eta}\Delta - 2\,\frac{\alpha_0^{-2N+\eta}}{\alpha_0 - 1}(1-\Delta) + X(y)$$

$$= 2\,\frac{\alpha_0^{-2N+\eta+1}}{\alpha_0 - 1}\Delta - 2\,\frac{\alpha_0^{-2N+\eta}}{\alpha_0 - 1} + X(y),$$

and that is less than $X(y)$, since $\Delta < 1/\alpha_0$. Q.E.D.

It now appears that we have rather hard estimates on the steps that the seeker can take to limit his losses in dealing with this problem. He can either accept an expected path length of $1+\alpha_0$ while risking a possible worst outcome of nearly $1+2\alpha_0^2/(\alpha_0-1)$, or else hold his maximal losses to 9; though to do so, he must accept an expected path length of $1+3/\ln 2$. Or he may compromise by choosing $2<\alpha<\alpha_0$, and playing the strategy indicated in Theorem 4, thus an expected loss of $1+(\alpha+1)/\ln\alpha$ and a maximal loss of no more than $1+2\alpha^2/(\alpha-1)$.

It might be, of course, that he would do very much better by making canny guess as to what the distribution F is likely to be. Whether or not this is true depends in part on one's definition of the words "much better".

We assume at the outset that the searcher has no knowledge, and this entitles us at least to the belief that he cannot distinguish between the ends of the line. Thus, if $\omega \to F_\omega$ is a mapping from some probability space Ω into the set of all probability distributions, then we can legitimately define our probability distribution over $\Omega\times\{+,-\}$, where $\{+,-\}$ is a random space, and

$$F_{(\omega,+)}(t) = F_\omega(t);\ F_{(\omega,-)}(t) = F_\omega(-t).$$

The mapping $\omega \to F_\omega$ must be chosen at least measurable enough so that the function $\int_{-\infty}^{+\infty} X(x,t)dF_\omega(t)$ is a measurable function of ω for each search procedure x, and we will assume that the generalized function

$$X(x,t)F'_\omega$$

is measurable in t and ω together for each x. It follows that

$$X(x,t)F_{(\omega,\pm)}$$

is measurable in t, ω, and $\pm$ together, and since it is non-negative, Fubini's Theorem applies.

Thus,

$$\int_{\Omega\times\{+,-\}}\int_{-\infty}^{+\infty} X(x,t)dF_{\bar\omega}(t)\Pr d\bar\omega$$

$$= \int_{\Omega}\int_{-\infty}^{+\infty}\int_{+,-} X(x,t)dF_{\bar\omega}(t)\Pr d\bar\omega$$

$$\geqq \int_{\Omega}\int_{-\infty}^{+\infty} 2\,|\,t\,|\,dF_{\omega}(t)\Pr d\omega$$

$$= \int_{\Omega} 2M_1(F_{\omega})\Pr d\omega = 2,$$

since $M_1(F_{\omega}) \equiv 1$. Thus, no assessment of likely distributions will bring the seeker below an expected path length of 2 unless it correctly guesses which is the favored end of line. A similar analysis holds if there is no favored end. By comparison, the constants discovered are $9, 1 + 3/\ln 2$ $(= 5.33)$, $1 + 2\alpha_0^2/(\alpha_0 - 1)$ $(= 10.90)$, and $1 + \alpha_0$ $(= 4.59)$. The reader may judge for himself whether the indicated courses of action represent a practical solution to the problem.

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